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Adiabatic switching for time-dependent electric fields

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In this work the scattering theory associated with the differential equation $i(\partial\psi/\partial t) = (-\Delta + e^{-\varepsilon|t|}g(t)x_1 + q(x))\psi$ is considered, where $x = (x_1,x^\perp)\in\mathbb{R}\times\mathbb{R}^2$, $\varepsilon\geqslant 0$, $\omega > 0$, $\alpha\in\mathbb{R}$, $g(t)$, $t\in\mathbb{R}$ is continuous, periodic with mean value zero over a period, and $q(x)$ approaches to zero sufficiently fast as $|x|\to\infty$. In the case $\varepsilon > 0$, it is shown that the usual theory is adequate; however, a limit does not exist when $\varepsilon\downarrow 0$. A modified theory is developed where the limit does exist as $\varepsilon\downarrow 0$. Furthermore, the concepts of bound states and scattering states for $\varepsilon\geqslant 0$ are discussed.

I. INTRODUCTION

In this paper we will discuss the scattering theory associated with the Cauchy problem,

$$\begin{aligned} &i\,\partial_t\psi = (-\Delta + e^{-\varepsilon|t|}g(t)x_1 + q(x))\psi,\\ &\psi(x,s) = \psi_s(x)\in L^2(\mathbb{R}^3), \end{aligned} \tag{1.1}$$

where $x = (x_1,x^\perp)\in\mathbb{R}\times\mathbb{R}^2$, t, $s\in\mathbb{R}$, $\varepsilon\geqslant 0$, $g(t)$, $q(x)$ are both real valued, $g(t)$ is continuous and bounded, and $q(x)$ has the form

$$\begin{aligned} &q(x) = q_1(x) + q_2(x),\\ &q_1\in L^\infty_\infty(\mathbb{R}^3),\quad q_2\in L^2(\mathbb{R}^3). \end{aligned} \tag{1.2}$$

Here $L^\infty_\infty(\mathbb{R}^3)$ denotes the set of $f\in L^\infty(\mathbb{R}^3)$ that tend to zero at infinity. Further assumptions on q and g will be introduced as we proceed. Under these conditions, the operator defined by

$$\begin{aligned} {\circ}A^\varepsilon(t) = -\Delta + e^{-\varepsilon|t|}g(t)x_1 + q(x),\quad &D({\circ}A^\varepsilon(t))\\ = C_0^\infty(\mathbb{R}^3),& \end{aligned} \tag{1.3}$$

is essentially self-adjoint (see Sec. 3 of Ref. 1 and references therein). We denote its closure by $A^\varepsilon(t)$ and write $A_0^\varepsilon(t)$ for the case $q=0$. As is well known, (1.1) describes the interaction of a quantum-mechanical particle in the semiclassical approximation with a potential $q(x)$ and the electric field $e^{-\varepsilon|t|}g(t)$ (1,0,0). The case $\varepsilon = 0$ was studied in Ref. 1 where existence and uniqueness of solutions for (1.1) was proved assuming that q_1 is also continuous. As pointed out by Kato[2] this assumption is not needed. It should be stressed, however, that the hypotheses in Ref. 1 already cover the Coulomb potential case, as far as existence and uniqueness are concerned. From now on we will assume that $g(t)$ is periodic with period $\tau > 0$ and

$$\int_0^\tau g(t)dt = 0. \tag{1.4}$$

In this case a satisfactory scattering theory was established in Ref. 1 (see also Ref. 3) under the assumptions

$$q(x) = (1 + |x|^2)^{-\rho}(W_1(x) + W_2(x)), \tag{1.5}$$

$$\rho > \tfrac{1}{2},\quad W_1\in L^\infty(\mathbb{R}^3),\quad W_2\in L^2(\mathbb{R}^3),$$

$$\frac{\partial W_1}{\partial x_1}\in L^2(\mathbb{R}^3), \tag{1.6}$$

where the derivative in (1.6) is computed in the sense of distributions. More precisely, if $U_{A^0}(t,s)$ is the propagator associated to (1.1) (with $\varepsilon = 0$) and $\Theta(s) = U_{A^0}(s+\tau,s)$ is the Floquet operator of the system, then

$$L^2(\mathbb{R}^3) = \mathscr{H}_{\rm ac}(\Theta(s))\oplus\mathscr{H}_p(\Theta(s)), \tag{1.7}$$

$$\mathscr{R}(\Omega_\pm(A^0,A_0^0;s)) = \mathscr{H}_{\rm ac}(\Theta(s)), \tag{1.8}$$

where $\mathscr{H}_p(U)$ and $\mathscr{H}_{\rm ac}(U)$ are, respectively, the pure point and absolutely continuous subspaces associated with the unitary operator U, and the wave operators are defined by

$$\Omega_\pm(A^0,A_0^0;s) = \operatorname*{s-lim}_{t\to\pm\infty} U_{A^0}(t,s)^* U_{A_0^0}(t,s). \tag{1.9}$$

It can also be shown[1] that $\mathscr{H}_p(\Theta(s))$ and $\mathscr{H}_{\rm ac}(\Theta(s))$ are precisely the bound state and scattering state subspaces in the time-dependent sense (see Sec. IV). In particular the "free" dynamics in this formulation is determined by the Hamiltonian $A_0^0(t)$. Although this is a very pleasing theory from the mathematical point of view, physically one would expect to be able to compare the dynamics generated by $A(t)$ with the one determined by $H_0 = -\Delta$ [the Laplacian in $L^2(\mathbb{R}^3)$] since, after all, the mean value of $A^0(t)$ over a period is simply $H = H_0 + q$ and there is a very well established scattering theory for the pair (H,H_0). That this can in fact be done by suitably modifying the wave operators is shown in Sec. IV. This was one of the main motivations for this work.

We were also interested in the so-called adiabatic switching of the field which is often used in physics (see Refs. 4–6 and the references therein). Roughly speaking, this procedure consists in introducing a "regularizing factor" depending continuously on some parameter $\varepsilon > 0$ (in our case $e^{-\varepsilon|t|}$), developing the theory in this situation and taking limits as $\varepsilon\downarrow 0$ in the hope of being able to handle the (in principle) more difficult case $\varepsilon = 0$. In connection with this, one should note that Dollard[7] has studied adiabatic switching in the usual theory of scattering. More precisely, he introduces the Hamiltonian $H(t) = H_0 + e^{-\varepsilon|t|}q$ and shows

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that if $q(x)$ is a short range potential, the usual wave operators with $\varepsilon>0$ exist and are unitary and, in the limit, they coincide with the wave operators for the pair (H,H_0). On the other hand, if q is the Coulomb potential the same result holds in the case $\varepsilon>0$, but the limit does not exist. Dollard also shows how to modify the theory in order to obtain the right wave operators as $\varepsilon\downarrow 0$. Note that in both situations there are no bound states if $\varepsilon>0$. In the electric field case the situation is different. In Sec. III we show that if $\varepsilon>0$ and $H=H_0+q$ has a bound state then there are solutions of (1.1) that behave as bound states as $t\to\pm\infty$. We also prove that the usual wave operators exist. In the following section we show that these operators do not have a limit as $\varepsilon\downarrow 0$. The definitions are then modified and a satisfactory scattering theory is obtained in the limit, as mentioned above. Section II contains some notation and various technical results that will be used in the remainder of this work.

II. PRELIMINARIES

We begin by introducing several auxiliary functions which will be needed in the next three sections. Assume that $g:\mathbb{R}\to\mathbb{R}$ is continuous periodic with period $\tau>0$ and satisfies (1.4). In this case it is easy to see that we can choose h and G such that for all $t\in\mathbb{R}$,

$$\begin{aligned} &h'(t)=g(t), \quad G'(t)=h(t),\\ &h(t+\tau)=h(t), \quad G'(t+\tau)=G(t), \qquad (2.1)\\ &\int_0^\tau h(t)dt=\int_0^\tau G(t)dt=0. \end{aligned}$$

Moreover, we will also need $k(t)$ such that

$$k'(t)=h(t)^2. \qquad (2.2)$$

Next, if $\varepsilon\geqslant 0$, we define functions g^ε, h^ε, G^ε, and k^ε as follows. If $\varepsilon=0$ let g^0, h^0, G^0, k^0 be the functions just introduced. If $\varepsilon>0$, choose

$$g^\varepsilon(t)=\exp(-\varepsilon|t|)g(t), \qquad (2.3)$$

$$h^\varepsilon(t)=\begin{cases} -\int_t^\infty g^\varepsilon(s)ds, & t\geqslant 0,\\ \int_{-\infty}^t g^\varepsilon(s)ds, & t<0, \end{cases} \qquad (2.4)$$

$$G^\varepsilon(t)=\begin{cases} -\int_t^\infty h^\varepsilon(s)ds, & t\geqslant 0,\\ \int_{-\infty}^t h^\varepsilon(s)ds, & t<0, \end{cases} \qquad (2.5)$$

$$k^\varepsilon(t)=\begin{cases} -\int_t^\infty (h^\varepsilon(s))^2\,ds, & t\geqslant 0,\\ \int_{-\infty}^t (h^\varepsilon(s))^2\,ds, & t<0. \end{cases} \qquad (2.6)$$

Now assume that $q(x)$ satisfies (1.2) and let $\psi(x,t)$ be the solution of (1.1) with $\varepsilon\geqslant 0$ fixed (which exists globally and is unique; see Theorem 2.1 below), and introduce

$$\varphi(x,t)=\exp(ih^\varepsilon(t)x_1)\psi(x,t), \qquad (2.7)$$

$$\chi(x,t)=\exp(ik^\varepsilon(t))\varphi(x_1-2G^\varepsilon(t),x^\perp). \qquad (2.8)$$

Then an easy computation shows that φ and χ are solutions of the equations

$$i\,\partial_t\varphi=[((1/i)\partial_{x_1}-h^\varepsilon(t))^2-\Delta^\perp]\varphi+q\varphi, \qquad (2.9)$$

$$i\,\partial_t\chi=(-\Delta+q(x_1-2G^\varepsilon(t),x^\perp))\chi, \qquad (2.10)$$

where $\Delta^\perp$ denotes the Laplacian with respect to the $x^\perp$ variable.

Let $\circ A^\varepsilon(t)$, $\circ B^\varepsilon(t)$, and $\circ H^\varepsilon(t)$, be the Hamiltonians that occur on the right-hand sides of (1.1), (2.9), and (2.10) with domain $C_0^\infty(\mathbb{R}^3)$. These operators are essentially self-adjoint and we will denote their self-adjoint realizations in $L^2(\mathbb{R}_3)$ by $A^\varepsilon(t)$, $B^\varepsilon(t)$, and $H^\varepsilon(t)$ (see Ref. 1 and the references therein). In case $q=0$ we will write $A_0^\varepsilon(t)$, $B_0^\varepsilon(t)$, and H_0. Applying Kato's theory of existence and uniqueness for linear "hyperbolic" evolution equations,[8] it was shown in Ref. 1 that the following theorem holds.

Theorem 2.1: Let $K(t)$ denote any one of the three operators $A^\varepsilon(t)$, $B^\varepsilon(t)$, $H^\varepsilon(t)$. Then there exists a unique evolution operator (propagator) $U_K(t,s)$, $(t,s)\in\mathbb{R}^2$, solving

$$i\frac{d\theta}{dt}=K(t)\theta(t), \quad \theta(s)=\theta_s\in Y, \qquad (2.11)$$

where

$$Y=\{f\in L^2(\mathbb{R}^3)|\Delta f,(1+x_1^2)^{1/2}f\in L^2(\mathbb{R}^3)\} \qquad (2.12)$$

in the case of (1.1) and $Y=D(H_0)=H^2(\mathbb{R}^3)$ for the other two equations. Moreover

$$U_K(t,s)(Y)\subseteq Y \qquad (2.13)$$

in all three cases and the propagators are related by

$$\begin{aligned} U_{A^\varepsilon}(t,s)&=T^\varepsilon(t)^{-1}U_{B^\varepsilon}(t,s)T^\varepsilon(s)\\ &=T^\varepsilon(t)^{-1}V^\varepsilon(t)^{-1}U_{H^\varepsilon}(t,s)V^\varepsilon(s)T^\varepsilon(s), \qquad (2.14) \end{aligned}$$

with $T^\varepsilon(t)$, $V^\varepsilon(t)$, $t\in\mathbb{R}$, given by

$$(T^\varepsilon(t)f)(x)=\exp(ih^\varepsilon(t)x_1)f(x), \qquad (2.15)$$

$$(V^\varepsilon(t)f)(x)=\exp(ik^\varepsilon(t))f(x_1-2G^\varepsilon(t),x^\perp), \qquad (2.16)$$

for all $f\in L^2(\mathbb{R}^3)$.

Finally in the remainder of this paper we will need the following limiting properties of the auxiliary functions introduced at the beginning of this section.

Lemma 2.2: Let g, h, G, k, g^ε, h^ε, G^ε, k^ε be as above. Then, (i) for each fixed $\varepsilon>0$ we have

$$\lim_{t\to\pm\infty}h^\varepsilon(t)=\lim_{t\to\pm\infty}G^\varepsilon(t)=\lim_{t\to\pm\infty}k^\varepsilon(t)=0; \qquad (2.17)$$

(ii) for each fixed $t\in\mathbb{R}$, we have

$$\lim_{\varepsilon\downarrow 0}h^\varepsilon(t)=h(t), \quad \lim_{\varepsilon\downarrow 0}G^\varepsilon(t)=G(t), \qquad (2.18)$$

$$\lim_{\varepsilon\downarrow 0}(k^\varepsilon(t)-k^\varepsilon(s))=k(t)-k(s), \qquad (2.19)$$

$$\lim_{\varepsilon\downarrow 0}k^\varepsilon(t)=\begin{cases} -\infty, & \text{if} \quad t>0,\\ \infty, & \text{if} \quad t<0. \end{cases} \qquad (2.20)$$

Proof: We will concentrate on the case $t\geqslant 0$. Similar arguments hold for $t<0$. The limits in (2.17) and (2.18) follow by combining (2.3) and (2.4) in order to obtain the estimate

$$|h^\varepsilon(t)|\leqslant\varepsilon^{-1}e^{-\varepsilon t}\|g\|_\infty, \quad \forall t\geqslant 0, \qquad (2.21)$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm.

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Next, using $h' = g$, $G' = h$, and integrating by parts twice we obtain

$$\begin{aligned} h^{\varepsilon}(t) &= -\int_t^{\infty} \exp(-\varepsilon s) h'(s) ds \\ &= e^{-\varepsilon t} h(t) + \varepsilon \int_t^{\infty} \exp(-\varepsilon s) G'(s) ds \\ &= e^{-\varepsilon t} h(t) + \varepsilon e^{-\varepsilon t} G(t) \\ &\quad + \varepsilon^2 \int_t^{\infty} \exp(-\varepsilon s) G(s) ds \\ &= e^{-\varepsilon t} h(t) + \varepsilon e^{-\varepsilon t} G(t) \\ &\quad + \varepsilon \int_{\varepsilon t}^{\infty} e^{-\theta} G\left(\frac{\theta}{\varepsilon}\right) d\theta. \end{aligned} \tag{2.22}$$

Since G is a bounded function, the integral in the last member of (2.22) can be estimated by $\|G\|_{\infty} \exp(-\varepsilon t)$ and the first limit in (2.18) follows at once. In order to prove the second, note that since $G' = h$ the fourth equality in (2.22) implies

$$\begin{aligned} G^{\varepsilon}(t) &= e^{-\varepsilon t} G(t) + \varepsilon^2 \int_t^{\infty} ds \int_s^{\infty} du \exp(-\varepsilon u) G(u) \\ &= e^{-\varepsilon t} G(t) + \varepsilon^2 \int_t^{\infty} du (u-t) \exp(-\varepsilon u) G(u), \end{aligned} \tag{2.23}$$

and the result follows in the same way as the previous one. The only difference is that to control the integral of $u \exp(-\varepsilon u) G(u)$, we must use another function H, periodic with mean value zero such that $H' = G$, and integrate by parts in order to get the factor ε^3 where we need it. Equation (2.19) is an easy consequence of the dominated convergence theorem. We now turn to (2.20), which is by far the hardest part. From the third equality in (2.22) we get

$$\begin{aligned} h^{\varepsilon}(t)^2 &= e^{-2\varepsilon t} h(t)^2 + 2\varepsilon e^{-\varepsilon t} h(t) \int_t^{\infty} \exp(-\varepsilon s) h(s) ds \\ &\quad + \varepsilon^2 \left(\int_t^{\infty} \exp(-\varepsilon s) h(s) ds \right) \\ &\quad \times \left(\int_t^{\infty} \exp(-\varepsilon u) h(u) du \right). \end{aligned} \tag{2.24}$$

It is easy to see that after integration the last two terms of the right-hand side of (2.24) tend to zero as $\varepsilon \downarrow 0$. Thus it remains to show that

$$\int_t^{\infty} e^{-2\varepsilon s} (h(s))^2 ds \to \infty, \quad \text{as } \varepsilon \downarrow 0. \tag{2.25}$$

To do this let $\alpha = \sup_{s \in \mathbf{R}} h(s)^2$ and write

$$\begin{aligned} X_e &= \{ s \in [t, \infty) \mid h(s)^2 < \alpha/2 \}, \\ X_r &= \{ s \in [t, \infty) \mid h(s)^2 \geqslant \alpha/2 \}, \end{aligned} \tag{2.26}$$

so that $X_e \cap X_r$ is empty and $[t, \infty) = X_e \cup X_r$. Then,

$$\int_t^{\infty} e^{-\varepsilon s} (h(s))^2 ds \geqslant \int_{X_r} e^{-\varepsilon s} (h(s))^2 ds \geqslant \frac{\alpha}{2} \int_{X_r} e^{-\varepsilon s} ds. \tag{2.27}$$

But as $\varepsilon \downarrow 0$ the integral in the right-hand side of (2.27) tends to the Lesbegue measure $|X_r|$ of X_r, which is infinite since $(h(s))^2$ is periodic, non-negative, and nontrivial. This completes the proof. Q.E.D.

III. SCATTERING THEORY WITH $\varepsilon > 0$

The purpose of this section is to relate the asymptotic behavior of $U_{A^{\varepsilon}}(t,s)$ and $\exp(-i(t-s)H)$ as $t \to \pm \infty$, with both $\varepsilon > 0$ and $s \in \mathbf{R}$ kept fixed. In order to accomplish this it is convenient to establish a series of preliminary results, the first one of which is the following theorem.

Theorem 3.1: Assume that $q(x)$ satisfies

$$\begin{aligned} q(x) = (1 + |x|^2)^{-\rho} q_1(x) + q_2(x), \quad \rho > \tfrac{1}{2}, \\ q_1 \in L^{\infty}(\mathbf{R}^3), \quad q_2 \in L^2(\mathbf{R}^3). \end{aligned} \tag{3.1}$$

Then the wave operators

$$\Omega_{\pm}^{\varepsilon}(A^{\varepsilon}, A_0^{\varepsilon}; s) = \operatorname*{s-lim}_{t \to \pm \infty} U_{A^{\varepsilon}}(t,s)^* U_{A_0^{\varepsilon}}(t,s) \tag{3.2}$$

exist, where the right-hand side of (3.2) denotes, as usual, the strong limit in $L^2(\mathbf{R}^3)$.

Proof: We will consider only the limit as $t \to \infty$. The existence of the other follows from similar arguments. Furthermore, since the main idea involved here, namely the Cook–Kuroda method, is by now standard, we will just indicate the estimates involved. Note that in order to prove (3.2) (with $t \to +\infty$), it is enough to show that

$$\int_a^{\infty} \| q U_{A_0^{\varepsilon}}(t,s) \phi \|_{L^2} dt < \infty, \tag{3.3}$$

for some $a > s$ (which is fixed) and all $\phi \in C_0^{\infty}(\mathbf{R}^3)$. To do this we use (2.14)–(2.16) to write $U_{A_0^{\varepsilon}}(t,s)$ in terms of $\exp(-i(t-s)H_0)$ as follows:

$$\begin{aligned} U_{A_0^{\varepsilon}}(t,s) &= \exp[-i(h^{\varepsilon}(t) x_1 + k^{\varepsilon}(t))] \\ &\quad \times \exp(-i(t-s)H_0) S_{2G^{\varepsilon}(t) - 2G^{\varepsilon}(s)} \\ &\quad \times \exp(i h^{\varepsilon}(s)(x_1)). \end{aligned} \tag{3.4}$$

If $q \in L^2(\mathbf{R}^3)$, (3.4) implies that

$$|q(x)(U_{A_0^{\varepsilon}}(t,s)\phi)(x)| \leqslant C |t-s|^{-3/2} \|\phi\|_{L^1} |q(x)|, \tag{3.5}$$

where $C > 0$ is a constant. Integrating this inequality over $\mathbf{R}^3$, we obtain the estimate needed to prove (2.3) in this case. Next if $q(x) = (1 + |x|^2)^{-\rho} q_1(x)$, $q_1 \in L^{\infty}(\mathbf{R}^3)$, it is enough to consider $\frac{1}{2} < \rho < \frac{3}{4}$, since otherwise $q \in L^2(\mathbf{R}^3)$ and there is nothing to prove in view of the previous remarks. Using the fact that the first factor on the right-hand side of (3.4) commutes with multiplications and choosing p, $\tilde{p}$, r such that $r \in (3/2\rho, 3)$, $\tilde{p}^{-1} + r^{-1} = 2^{-1}$, $p^{-1} + \tilde{p}^{-1} = 1$, we can apply the Riesz–Thorin theorem[9] to conclude

$$\| q U_{A_0^{\varepsilon}}(t,s) \phi \|_2 \leqslant C \|q_1\|_{L^{\infty}} |t-s|^{-3/r} \|\phi\|_p, \tag{3.6}$$

where $C > 0$ is again a constant. Since $3/r > 1$, the result follows also in this case and the proof is complete. Q.E.D.

Note that the estimates in (3.5) and (3.6) are independent of ε, and therefore they also hold in the case $\varepsilon = 0$ [in particular the wave operators $\Omega_{\pm}(A^0, A_0^0; s)$ exist under the assumptions made in Theorem 3.1; this result is stronger than the corresponding existence theorem in Sec. 5 of Ref. 1]. This remark will be used in Sec. IV.

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Next we prove a technical lemma that will be useful in the remainder of this section and has some interest of its own.

Lemma 3.2: Assume that $q(x)$ satisfies (1.5) and let $H = H_0 + q$. Then

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{i(t-s)H}T^\varepsilon(t)e^{-i(t-s)H} = 1, \tag{3.7}$$

for all $s\in\mathbb{R}$, where 1 denotes the identity operator in $L^2(\mathbb{R}^3)$.

Proof: Without loss of generality we will assume that $s = 0$. It is well known that under assumption (1.5) the following decomposition holds[10,11]:

$$L^2(\mathbb{R}^3) = \mathscr{H}_p(H) \oplus \mathscr{H}_{\rm ac}(H), \tag{3.8}$$

where $\mathscr{H}_p(H)$ [resp. $\mathscr{H}_{\rm ac}(H)$] denotes the pure point (resp. absolutely continuous) subspace of $L^2(\mathbb{R}^3)$ with respect to H (for the definition of these objects see Refs. 11 or 12). In order to prove the results we will show that the limit exists in the two subspaces on the right-hand side of (3.8). We start with $\mathscr{H}_p(H)$. Let $f\in\mathscr{D}(H) = \mathscr{D}(H_0)$ be such that $Hf = \lambda f$ for some $\lambda\in\mathbb{R}$. Then

$$\|e^{itH}T^\varepsilon(t)e^{-itH}f - f\|_{L^2} = \|T^\varepsilon(t)f - f\|_{L^2}, \tag{3.9}$$

and the right-hand side of (3.9) tends to zero as $t\to\infty$, since by the dominated convergence theorem we have

$$\operatorname*{s-lim}_{t\to\pm\infty} T^\varepsilon(t) = 1. \tag{3.10}$$

Using a simple approximation argument we obtain the result in $\mathscr{H}_p(H)$. Next we turn to $\mathscr{H}_{\rm ac}(H)$. We will consider only $t\to+\infty$. The other case can be treated similarly. Recall from usual potential scattering that given $f\in\mathscr{H}_{\rm ac}(H)$ there exists a unique $\varphi_+\in L^2(\mathbb{R}^3)$ such that

$$\lim_{t\to\infty} \|e^{-itH}f - e^{-itH_0}\varphi_+\|_{L^2} = 0. \tag{3.11}$$

For a proof of this statement we refer the reader to Ref. 10 and/or Ref. 11. Adding and subtracting the appropriate quantities, using the triangle inequality and the unitarity of e^{itH} and $T^\varepsilon(t)$ for each $t\in\mathbb{R}$, we obtain

$$\|e^{itH}T^\varepsilon(t)e^{-itH}f - f\|_{L^2} \leqslant 2\|e^{-itH}f - e^{-itH_0}\varphi_+\|_{L^2} + \|(T^\varepsilon(t) - 1)e^{-itH_0}\varphi_+\|_{L^2}. \tag{3.12}$$

In view of (3.11) it remains to show that the second term on the right-hand side of (3.12) tends to zero as $t\to\infty$. Let $\hat\theta$ denote the Fourier transform of $\theta\in L^2(\mathbb{R}^3)$ (for details, see Ref. 9). Given $\delta > 0$ choose $\hat\theta\in C_0^\infty(\mathbb{R}^3)$ such that $\|\theta - \varphi_+\|_{L^2} < \delta$. Then

$$\|(T^\varepsilon(t) - 1)e^{-itH_0}\varphi_+\|_{L^2}^2 \leqslant (\|(T^\varepsilon(t) - 1)e^{-itH_0}\theta\|_{L^2} + \delta)^2, \tag{3.13}$$

so that it suffices to prove that $\|(T^\varepsilon(t) - 1)e^{-itH_0}\theta\|_{L^2}$ tends to zero as $t\to\infty$. Applying Parseval's identity[9] we obtain

$$\|(T^\varepsilon(t) - 1)e^{-itH_0}\theta\|_{L^2}^2 = \int_{\mathbb{R}^3} |E(\xi_1 - h^\varepsilon(t),\xi^\perp,t) - E(\xi,t)|^2\, d\xi, \tag{3.14}$$

where $E(\xi,t) = \exp(-it\xi^2)\hat\theta(\xi)$. Since $h^\varepsilon(t)$ is a bounded function, it is easy to see that the integrand in the rhs of (3.14) has compact support. But then the dominated convergence theorem implies the result because according to (2.21) both $h^\varepsilon(t)$ and $t\,h^\varepsilon(t)$ tend to zero as $t\to\infty$. Q.E.D.

As a consequence of Lemma 3.2 we obtain the following important result which relates the asymptotic behaviors of the propagators $U_{A_0^\varepsilon}(t,s)$ and $\exp(-i(t-s)H_0)$.

Corollary 3.3: Let $U_{A_0^\varepsilon}(t,s)$ and H_0 be as in Theorem 2.1 (with $q = 0$). Then

$$\begin{aligned}\Omega^\varepsilon_\pm(A_0^\varepsilon,H_0;s) &= \operatorname*{s-lim}_{t\to\pm\infty} U_{A_0^\varepsilon}(t,s)^* e^{-i(t-s)H_0} \\ &= V^\varepsilon(s)^{-1}T^\varepsilon(s)^{-1} \\ &= \exp(-ik^\varepsilon(s))\exp(-ih^\varepsilon(s)x_1)S_{2G^\varepsilon(s)},\end{aligned} \tag{3.15}$$

where S_a, $a\in\mathbb{R}$, is given by

$$(S_a f)(x) = f(x_1 + a,x^\perp), \quad f\in L^2(\mathbb{R}^3). \tag{3.16}$$

In particular the operators $\Omega_\pm(A_0^\varepsilon,H_0;s)$ are unitary.

Proof: Applying (2.14) with $q = 0$ and noting that $V^\varepsilon(t)$ commutes with $\exp(-i(t-s)H_0)$ we obtain

$$U_{A_0^\varepsilon}(t,s)^*\exp(-i(t-s)H_0) = T^\varepsilon(s)^{-1}V^\varepsilon(s)^{-1}V^\varepsilon(t)e^{i(t-s)H_0}T^\varepsilon(t)e^{-i(t-s)H_0}, \tag{3.17}$$

and the result follows at once from Lemma 3.2 and part (i) of Lemma 2.2 which implies that $\operatorname{s-lim}_{t\to\pm\infty} V^\varepsilon(t) = 1$. Q.E.D.

Combining Theorem 3.1 and Corollary 3.3 it is easy to prove the following Corollary.

Corollary 3.4: Let q be as in Theorem 3.1. Then the wave operators

$$\Omega^\varepsilon_\pm(A^\varepsilon,H_0;s) = \operatorname*{s-lim}_{t\to\pm\infty} U_{A^\varepsilon}(t,s)^* e^{-i(t-s)H_0} \tag{3.18}$$

exist.

We are now in position to state and prove the main result of this section, namely, the following Theorem.

Theorem 3.5: Assume that q satisfies condition (1.5). Let $H = H_0 + q$ and $A^\varepsilon(t)$ be as in Theorem 2.1. Then the limits

$$\Gamma_\pm(A^\varepsilon,H;s) = \operatorname*{s-lim}_{t\to\pm\infty} U_{A^\varepsilon}(t,s)^* e^{-i(t-s)H} \tag{3.19}$$

exist and are unitary. Moreover the following intertwining relations holds:

$$U_{A^\varepsilon}(t,s)\Gamma_\pm(A^\varepsilon,H;s) = \Gamma_\pm(A^\varepsilon,H;t)e^{-i(t-s)H}. \tag{3.20}$$

Proof: In view of the first equality in (2.14) we may write $\Gamma(t) = U_{A^\varepsilon}(t,s)^* e^{-i(t-s)H}$ as

$$T(t) = T^\varepsilon(s)^{-1}(U_{B^\varepsilon}(t,s)^* e^{-i(t-s)H}) \times e^{i(t-s)H}T^\varepsilon(t)e^{-i(t-s)H}, \tag{3.21}$$

where $B^\varepsilon(t)$ is as in Theorem 2.1. Due to Lemma 3.2 [and the uniform boundness of all the factors in (3.21) with respect to t], it is enough to show that the limits

$$\Gamma_\pm(B^\varepsilon,H;s) = \operatorname*{s-lim}_{t\to\pm\infty} U_{B^\varepsilon}(t,s)^* e^{-i(t-s)H} \tag{3.22}$$

exist and are unitary, since, in this case,

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$$\lim_{t\to\pm\infty}\Gamma(t)=\Gamma_\pm(A^\varepsilon,H;s)=T^\varepsilon(s)^{-1}\Gamma_\pm(B^\varepsilon,H;s),\tag{3.23}$$

which are obviously unitary. In order to obtain (3.22), we remark first that, as is well known (Sec. 3 of Ref. 1), we have

$$D(B^\varepsilon(t))=D(H)=D(H_0),\tag{3.24}$$

for all $t\in\mathbb{R}$. Let $\mathscr{G}$ denote $D(H_0)$ provided with the graph norm $|\,\|f\|\,|=\|f\|^2_{L^2}+\|H_0f\|^2_{L^2}$ and let $\mathscr{B}=B(\mathscr{G},L^2(\mathbb{R}^3))$ denote the set of all bounded operators from $\mathscr{G}$ into $L^2(\mathbb{R}^3)$. Then it is easy to verify that

$$\int_{\mathbb{R}}\|B^\varepsilon(t)-H\|_{\mathscr{B}}\,dt<\infty,\tag{3.25}$$

$$\operatorname*{Var}_{\mathbb{R}}(B(\cdot))=\sup\sum_{0\leqslant j\leqslant n-1}\|B^\varepsilon(t_{j+1})-B^\varepsilon(t_j)\|_{\mathscr{B}}<\infty,\tag{3.26}$$

where the supremun is taken over all finite real sequences $t_0<t_1<t_2<\cdots<t_n$. Under these conditions Theorem 6 of Ref. 13 implies that the operators in (3.22) exist and have the stated properties. The proof of the intertwining relations is standard and will be omitted (see Chap. X of Ref. 12, where the proof is presented in the case of time-independent Hamiltonians; the same idea works in our case). Q.E.D.

A few remarks are now in order. Let $\varphi\in\mathscr{H}_p(H)$. Then if $f_\pm=\Gamma_\pm(A^\varepsilon,H;s)\varphi$, we have

$$\lim_{t\to\pm\infty}\|U_{A^\varepsilon}(t,s)f_\pm-e^{-i(t-s)H}\varphi\|_{L^2}=0,\tag{3.27}$$

and it is easy to see that the wave functions $\psi_\pm(t)=U_{A^\varepsilon}(t,s)f_\pm$ behave as bound states as $t\to\pm\infty$. More precisely, the probability of finding the particle in $\{|x|\geqslant R\}$ at time t can be estimated as follows:

$$\begin{aligned}P(t,\{|x|\geqslant R\};f_\pm)&=\|f_\pm\|^{-2}\int_{\mathbb{R}^3}|\chi_{\{|x|>R\}}(x)\\&\quad\times(U_{A^\varepsilon}(t,s)f_\pm)(x)|^2\,dx\\&\leqslant\|f_\pm\|^{-2}(\|\chi_{\{|x|>R\}}e^{-i(t-s)H}\varphi\|\\&\quad+\|U_{A^\varepsilon}(t,s)f_\pm-e^{-i(t-s)H}\varphi\|)^2,\end{aligned}\tag{3.28}$$

where χ_s is the characteristic function of the set S. Thus, given $\eta>0$, there exist $t_0>0$ and $R_0>0$ such that if $|t|>t_0$ and $R>R_0$ then $P(t;\{|x|\geqslant R\};f_\pm)<\eta$. This means that the particle is asymptotically (as $t\to\pm\infty$) in a bound state. Moreover, it can also easily be shown that if $\varphi\in\mathscr{H}_{ac}(H)$, then $f_\pm=\Gamma_\pm(A^\varepsilon,H;s)\varphi$ are such that $\psi_\pm=U_{A^\varepsilon}(t,s)f_\pm$ behave as scattering states as $t\to\pm\infty$.

In view of the remarks just made, Theorem 3.5 and Eq. (3.8) imply two decompositions of $L^2(\mathbb{R}^3)$ into (asymptotic) bound state and scattering subspaces, namely,

$$\begin{aligned}L^2(\mathbb{R}^3)&=\Gamma_\pm(A^\varepsilon,H;s)(\mathscr{H}_{ac}(H))\\&\quad\oplus\Gamma_\pm(A^\varepsilon,H;s)(\mathscr{H}_p(H)).\end{aligned}$$

It should be remarked, however, that, as far as we know, it is an open question whether or not the above decompositions coincide.

IV. THE ADIABATIC LIMIT

In this section we will be concerned with the asymptotic behavior (in time) of the solution (1.1) as $\varepsilon\downarrow0$. The first thing to notice is that it is hopeless to take the "limit of the theory" established for $\varepsilon>0$. This is already apparent in Corollary 3.3. Indeed, in view of (3.15) and the behavior of $G^\varepsilon(s)$, $h^\varepsilon(s)$, and $k^\varepsilon(s)$, described in Lemma 2.2, it follows that $\Omega_\pm(A_0^\varepsilon,H_0;s)$ does not have a limit as $\varepsilon\downarrow0$. This also indicates what the problem is and points the way to the correct definitions. Let $\varepsilon\geqslant0$ and introduce

$$\begin{aligned}\Lambda^\varepsilon(t,s)&=e^{ik^\varepsilon(s)}T^\varepsilon(t)^{-1}V^\varepsilon(t)^{-1}\\&=\exp[-i(k^\varepsilon(t)-k^\varepsilon(s))]\\&\quad\times\exp[-ih^\varepsilon(t)x_1]S_{2G^\varepsilon(t)}.\end{aligned}\tag{4.1}$$

Define the modified wave operators for the pair $(A^\varepsilon(\cdot),H_0)$ by

$$W_\pm(A^\varepsilon,H_0;s)=\operatorname*{s-lim}_{t\to\pm\infty}U_{A^\varepsilon}(t,s)^*\Lambda^\varepsilon(t,s)e^{-i(t-s)H_0},\tag{4.2}$$

if the limits exist.

In what follows we will show that they indeed exist for $\varepsilon\geqslant0$ and that (4.2) is continuous in ε up to $\varepsilon=0$. We begin with the case $q=0$, which is trivial. Applying (2.15) to write $U_{A_0^\varepsilon}(t,s)$ in terms of $\exp(-i(t-s)H_0)$ and using the definition of $\Lambda^\varepsilon(t,s)$, we obtain

$$\begin{aligned}&U_{A_0^\varepsilon}(t,s)^*\Lambda^\varepsilon(t,s)e^{-i(t-s)H_0}\\&\quad=T^\varepsilon(s)^{-1}(e^{k^\varepsilon(s)}V^\varepsilon(s))^{-1}=T^\varepsilon(s)^{-1}S_{-2G^\varepsilon(s)},\end{aligned}\tag{4.3}$$

for all $\varepsilon\geqslant0$. Note that this expression is independent of t! This means that the modification just introduced cancels out the oscillations responsible for nonexistence of the limit of $\Omega_\pm(A_0^\varepsilon,H_0;s)$ as $\varepsilon\downarrow0$, uniformly in t. It should also be noted that $\Lambda^\varepsilon(t,s)e^{-i(t-s)H_0}$ is a "modified free evolution" in the sense that

$$\lim_{t\to\pm\infty}\int_S|\Lambda^\varepsilon(t,s)e^{-i(t-s)H_0}f(x)|^2\,dx=0,$$

for all bounded measurable $S\subseteq\mathbb{R}^3$ and $f\in L^2(\mathbb{R}^3)$.

In order to proceed, we will assume from now on that q satisfies (1.5) and (1.6). In this case, as shown in Sec. 5 of Ref. 1, the wave operators

$$\Omega_\pm(A^0,A_0^0;s)=\operatorname*{s-lim}_{t\to\pm\infty}U_{A^0}(t,s)^*U_{A_0^0}(t,s)\tag{4.4}$$

exist and are complete in the sense of (1.7) and (1.8) for all $s\in\mathbb{R}$, where $\Theta(s)$ is the Floquet (or period) operator of the system, namely,

$$\Theta(s)=U_{A^0}(s+\tau,s),\quad s\in\mathbb{R}.\tag{4.5}$$

With these remarks in mind, we have the following theorem.

Theorem 4.2: Let q satisfy (1.5) and (1.6). Then

$$\Omega_\pm(A^0,A_0^0;s)=\operatorname*{s-lim}_{\varepsilon\downarrow0}\Omega_\pm(A^\varepsilon,A_0^\varepsilon;s).\tag{4.6}$$

Proof: We will consider the case of $\Omega_+(A^0,A_0^0;s)$. The other limit can be handled similarly. Moreover, since all operators involved are uniformly bounded with respect to $\varepsilon\geqslant0$,

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it is enough to prove that the limit exists in $C_0^\infty(\mathbb{R}^3)$. Thus if φ is any such function, we have

$$\begin{aligned}\|\Omega_+(A^0,A_0^0;s)\varphi - \Omega_+(A^\varepsilon,A_0^\varepsilon;s)\varphi\|_{L^2} &\leqslant \|\Omega_+(A^0,A_0^0;s)\varphi \\ &- U_{A^0}(t,s)^* U_{A_0^0}(t,s)\varphi\|_{L^2} + \|U_{A^0}(t,s)^* U_{A_0^0}(t,s)\varphi \\ &- U_{A^\varepsilon}(t,s)^* U_{A_0^\varepsilon}(t,s)\varphi\|_{L^2} + \|U_{A^\varepsilon}(t,s)^* U_{A_0^\varepsilon}(t,s)\varphi \\ &- \Omega_+(A^\varepsilon,A_0^\varepsilon;s)\varphi\|_{L^2},\end{aligned} \tag{4.7}$$

for all $t\in\mathbb{R}$. According to the remark following the proof of Theorem 3.1, the first and third terms in the right-hand side of (4.7) can be estimated as follows:

$$\begin{aligned}&\|\Omega_+(A^\varepsilon,A_0^\varepsilon;s)\varphi - U_{A^\varepsilon}(t,s)^* U_{A_0^\varepsilon}(t,s)\varphi\|_{L^2} \\ &\quad\leqslant \int_t^\infty \|qU_{A_0^\varepsilon}(r,s)\varphi\|_{L^2}\,dr \\ &\quad\leqslant C\Big(\|q_1\|_{L^\infty}\|\varphi\|_p \int_t^\infty |u-s|^{-3/r}\,du \\ &\qquad + \|q_2\|_2\|\varphi\|_{L^1}\int_t^\infty |u-s|^{-3/2}\,du\Big),\end{aligned} \tag{4.8}$$

where $\varepsilon\geqslant 0$, $t>s$, and C is a constant independent of ε. Since the last member of (4.8) tends to zero as $t\to\infty$, it remains to show that the second term on the rhs of (4.7) tends to zero as $\varepsilon\downarrow 0$. In order to do this note that the differential equation satisfied by the propagators in question implies

$$\begin{aligned}U_{A^\varepsilon}(t,s)^*\varphi &= U_{A^0}(t,s)^*\varphi + i\int_s^t U_{A^\varepsilon}(r,s)^* \\ &\quad\times(e^{-\varepsilon r}-1)\cdot g(t)x_1 U_{A^0}(t,r)dr.\end{aligned} \tag{4.9}$$

Before proceeding it should be remarked that $x_1U_{A^0}(t,r)\varphi$ belongs to $L^2(\mathbb{R}^3)$ and depends continuously in t because of (2.13). Then

$$\begin{aligned}&\|U_{A^\varepsilon}(t,s)^*\varphi - U_{A^0}(t,s)^*\varphi\| \\ &\quad\leqslant \int_s^t |e^{-\varepsilon r}-1|\,\|x_1U_{A^0}(t,r)^*\varphi\|dr\end{aligned} \tag{4.10}$$

and the rhs tends to zero as $\varepsilon\downarrow 0$ by the dominated convergence theorem. This completes the proof. Q.E.D.

We now turn to the main result of this section, namely, the following theorem.

Theorem 4.3: Let q satisfy (1.5) and (1.6). Then the wave operators $W_\pm(A^\varepsilon,H_0;s)$ exist of all $\varepsilon\geqslant 0$. If $\varepsilon>0$, they are given by

$$W_+(A^\varepsilon,H_0;s) = \Omega_\pm(A^\varepsilon,A_0^\varepsilon;s)T^\varepsilon(s)^{-1}S_{-2G^\varepsilon(s)}, \tag{4.11}$$

while if $\varepsilon=0$, we have

$$\begin{aligned}W_\pm(A^0,H_0;s) &= \lim_{\varepsilon\downarrow 0} W_\pm(A^\varepsilon,A_0^\varepsilon;s) \\ &= \Omega_\pm(A^0,A_0^0;s)T(s)^{-1}S_{-2G(s)}.\end{aligned} \tag{4.12}$$

In particular,

$$\mathscr{R}(W_\pm(A^0,H_0;s)) = \mathscr{R}(\Omega_\pm(A^0,A_0^0;s)) = \mathscr{H}_{\rm ac}(\Theta(s)), \tag{4.13}$$

where $\Theta(s)$ is the Floquet operator defined in (4.5).

Proof: Due to (4.3), we can write

$$\begin{aligned}&U_{A^\varepsilon}(t,s)^*\Lambda^\varepsilon(t,s)e^{-i(t-s)H_0} \\ &\quad= U_{A^\varepsilon}(t,s)^*U_{A_0^\varepsilon}(t,s)T^\varepsilon(s)^{-1}S_{-2G^\varepsilon(s)},\end{aligned} \tag{4.14}$$

for all $t\in\mathbb{R}$ and $\varepsilon\geqslant 0$. Taking the limit as $t\to\pm\infty$, we obtain (4.11) and the second equality in (4.12). Next recall that in the proof of Theorem 4.2 we have shown that

$$\operatorname*{s-lim}_{\varepsilon\downarrow 0} U_{A^\varepsilon}(t,s)^*U_{A_0^\varepsilon}(t,s) = U_{A^0}(t,s)^*U_{A_0^0}(t,s)$$

[see the second term on the rhs of (4.7)]. Therefore

$$\begin{aligned}&\operatorname*{s-lim}_{t\to\pm\infty}\operatorname*{s-lim}_{\varepsilon\downarrow 0} U_{A^\varepsilon}(t,s)^*U_{A_0^\varepsilon}(t,s)T^\varepsilon(s)^{-1}S_{-2G^\varepsilon(s)} \\ &\quad= \operatorname*{s-lim}_{t\to\pm\infty} U_{A^0}(t,s)^*U_{A_0^0}(t,s)T(s)^{-1}S_{-2G(s)} \\ &\quad= \Omega_\pm(A^0,A_0^0;s)T(s)^{-1}S_{-2G(s)} = W_\pm(A^0,A_0^0;s),\end{aligned} \tag{4.15}$$

since we already know that the last equality in (4.15) holds. The statement about the Floquet operator and the ranges of the wave operators follows from (4.6) and the proof is complete. Q.E.D.

We will now make some final remarks on the results presented above. First of all, it is natural to ask what is the relation between the modified and usual theories when $\varepsilon>0$. The answer, which is not difficult to obtain, is given by the relation

$$\begin{aligned}\tilde{\Gamma}_\pm(A^\varepsilon,H;s) &= \operatorname*{s-lim}_{t\to\pm\infty} U_{A^\varepsilon}(t,s)^*\Lambda^\varepsilon(t,s)e^{-i(t-s)H} \\ &= e^{-ik^\varepsilon(s)}\Gamma_\pm(A^\varepsilon,H;s),\end{aligned} \tag{4.16}$$

where $\Gamma_\pm(A^\varepsilon,H;s)$ is the operator defined in Theorem 3.5. Thus we obtain two decompositions of $L^2(\mathbb{R}^3)$ into scattering and (asymptotic) bound states which are exactly the same as before except for a phase [which does not have a limit as $\varepsilon\downarrow 0$; see (2.20)]. In particular, we do not know if the decompositions coincide. In the limit, however, the results of this section show that we can construct a satisfactory scattering theory. In this case we may have bound states in the usual time-dependent sense[1,11] and the set of scattering states is exactly the same as those obtained in Ref. 1 using $U_{A^0}(t,s)$ as the free evolution.

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